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STOCHASTIC BOUNDARY ELEMENTS FOR TWO-DIMENSIONAL POTENTIAL FLOW IN HOMOGENEOUS DOMAINS

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Abstract—A stochastic boundary element formulation is presented for the analysis of two-dimensional steady state potential flow through homogeneous domains. The operator of the governing differential equation is assumed to be random and is described by a set of correlated random variables. The perturbation method, in conjunction with the boundary element method, is employed to derive the systems of equations for the unknown boundary variables and their respective first and second order derivatives with respect to the random variables. These quantities are then used to calculate the desired response statistics. A general procedure is developed which is next applied for the specific cases of random geometric configuration and random material parameter. The random geometric configuration is modeled using a finite set of correlated random variables. The random material parameter is modeled as a homogeneous random field which allows the use of deterministic fundamental solutions and integral representations for homogeneous domains. The random field is first discretized into a set of correlated random variables and then the general procedure is applied. A transformation of the correlated random variables into an uncorrelated set is performed to reduce the number of numerical operations. The results for the boundary variables are used to calculate the response statistics of internal potentials. These calculations require the modeling of the interior of the domain under consideration. Several models for representing the interior of the domain are presented for both random configuration and random material parameter and their influence on the response statistics is analysed. Distributed sources are considered in the present study using the particular integral approach. A number of numerical examples are presented to demonstrate the validity of the present formulations. The results obtained from the present analyses are compared with those obtained from Monte Carlo simulations with 5000 samples and a good agreement of results is observed.

1. INTRODUCTION

The randomness in the analysis of a physical problem may arise due to: (a) uncertain material and geometric parameters of the domain being analysed, and (b) random external influences that act on the object. Problems that involve random properties of the domain are referred to as parametric random problems, and those that involve random excitations are referred to as nonparametric problems (Benaroya and Rehak, 1988). Random properties of the domain result in a random operator of the differential equation that describes the problem. Only a few analytical solutions are available in the literature (Song, 1973; Adomain, 1983) for the solution of parametric random boundary value problems. Approximate methods are therefore required to be developed for the analysis of realistic problems arising in engineering practice. Most of the numerical techniques for the treatment of problems with random operator have been developed in the finite element framework. These include perturbation methods developed by Liu *et al.* (1986a, b, 1988), the basis random variable method (Lawrence, 1987) and the spectral decomposition method (Ghanem and Spanos, 1990). The research on probabilistic treatment of problems in structural mechanics has been summarized in textbooks, among others, those by Lin (1967) and Augusti *et al.* (1984). Current research trends and applications of probabilistic formulations to a variety of problems in structural engineering, such as reliability theory, stochastic finite element method, geotechnical engineering, fatigue reliability and stochastic fracture mechanics have been presented (Konishi *et al.*, 1985).

The boundary element method (Banerjee and Butterfield, 1981) possesses some inherent advantages over the finite element method that make it particularly attractive for the solution of probabilistic problems (Kaljević and Saigal, 1993). The application of the boundary element method for the solution of parametric boundary value problems, however, has been initiated only recently and only a few publications are available in the literature. Ettouney *et al.* (1989a, b) and Daddazio and Ettouney (1989) considered the material properties as random variables for two-dimensional problems in elastostatics and for radiation from a pulsating sphere, respectively. A general perturbation method, in conjunction with the boundary element method, was developed by Kaljević and Saigal (1993) for the treatment of two-dimensional problems in elastostatics that involve random domain parameters such as a random configuration of the domain or random material properties. The parameters defining the random configuration were modeled using correlated random variables, and that describing the random material was modeled using a homogeneous random field. The assumption of the homogeneous random field and the use of the perturbation method, where all quantities that depend on the random parameters are expanded in a Taylor series about the mathematical expectations of the random variables, allowed the use of the fundamental solutions for homogeneous domains in the analysis of problems with random material parameters. The systems of equations were derived for the response variables and their respective first and second order derivatives with respect to the random parameters. These quantities were later used to calculate the mathematical expectations and variances of the desired response variables. The derivatives of the boundary element kernels were obtained using analytical differentiation and the systems of equations obtained were solved using direct techniques.

The formulation described above is extended in this study for the treatment of problems of two-dimensional steady state potential flow through homogeneous domains. The randomness due to both a random geometric configuration of the domain and a random material description are studied. The parameters that describe the random contour of the object are modeled using a set of correlated random variables. The cases of circular and elliptical contours are analysed in detail. The derivatives of the boundary element kernels that are required in the analysis are calculated analytically. The strongly singular terms and their derivatives are calculated indirectly, using a uniform state of unit potential along the contour (Banerjee and Butterfield, 1981) and the implicit differentiation technique (Saigal *et al.*, 1989), respectively. It is shown that the derivatives of weakly singular terms do not introduce higher order singularities, which allows the use of standard numerical integration schemes in their calculations. The distributed sources within the domain of the object are treated using the particular integral approach (Pape and Banerjee, 1987; Henry and Banerjee, 1988). The particular integrals as well as their derivatives are derived for several cases of distributed sources. The domains characterized by a random material parameter are analysed next. It is shown that for certain boundary conditions, and in the absence of distributed sources, the random description of the material parameter does not affect the response of the object. When domain loads are present, the random material parameter is modeled using a homogeneous random field. The random field is first discretized into a set of correlated random variables defined for each boundary element, and the general procedure is applied to obtain the response statistics of the unknown contour variables. The transformation of correlated random variables into an uncorrelated set (Kaljević and Saigal, 1993) is performed to reduce the number of numerical operations. Only a small number of transformed random variables with the highest variances are retained in the analysis.

The developments concerning the response statistics of internal potentials are also presented in this paper. These calculations require the modeling of the interior of the domain. In the case of a random geometric configuration, the derivatives of the coordinates of internal points with respect to the random geometric parameters are required in order to calculate the derivatives of the internal potentials. Several schemes for modeling the interior of the domain are presented and their influence on the response statistics is analysed. It is shown that the assumptions on the randomness of the interior of the domain do not affect the response statistics of the boundary variables. In the case of a random material parameter, an additional random variable is introduced for each internal point. If the response statistics are required at a large number of internal points, a significant increase in the CPU time requirements may result. The use of transformed random variables in the analysis significantly reduces these additional CPU requirements.

Several numerical examples are solved and the results are compared with solutions obtained from Monte Carlo simulations to verify the accuracy of the formulations developed in this study.

2. DETERMINISTIC BOUNDARY VALUE PROBLEM

A review of the boundary element equations that describe deterministic potential flow through a homogeneous medium is first briefly presented to introduce the notation. The governing differential equation, its properties and the techniques employed for analytical solution may be found in standard mathematical textbooks (Flanigan, 1983; Hildebrand, 1972). Details of the boundary element formulation for a numerical solution of this equation may be found in, for example, the text by Banerjee and Butterfield (1981).

A steady state potential flow through a homogeneous domain, Ω, bounded by a contour, Γ, is described by the Poisson's equation, given as

$$k\nabla^2\Phi+\Psi = 0, \tag{1}$$

where Φ is the potential, Ψ represents distributed external sources and sinks, and k is the material parameter of the domain. Using Green's second identity (Flanigan, 1983), combined with the fundamental solution for an infinite medium, an integral representation for the potential, Φ, is obtained as

$$c(\xi)\Phi(\xi) = \int_\Gamma [G(\mathbf{x},\xi)\Phi_{,n}(\mathbf{x})-F(\mathbf{x},\xi)\Phi(\mathbf{x})]\,\mathrm{d}\Gamma(\mathbf{x})+\int_\Omega G(\mathbf{z},\xi)\Psi(\mathbf{z})\,\mathrm{d}\Omega(\mathbf{z}), \tag{2}$$

where $\Phi_{,n}(\mathbf{x})$ is the normal derivative of the potential, $\Phi(\mathbf{x})$; $c(\xi)$ is the jump term; $G(\mathbf{x},\xi)$ and $F(\mathbf{x},\xi)$ are boundary element kernels, given as

$$G(\mathbf{x},\xi) = -\frac{1}{2\pi}\log r \quad F(\mathbf{x},\xi) = -\frac{1}{2\pi}\frac{y_p n_p}{r^2}. \tag{3}$$

$\mathbf{x}$ and ξ denote two points on the contour, Γ, with respective coordinates (x_1, x_2) and (ξ_1, ξ_2); r is the distance between the points $\mathbf{x}$ and ξ; n_p, $p = 1,2$, are the components of the outward normal vector; and $y_p = x_p-\xi_p$. Equation (2) is discretized using boundary elements to obtain the system of equations for the boundary variables as

$$[F]\{\Phi\} = [G]\{\Phi_{,n}\}+\{\Psi\}, \tag{4}$$

where $\{\Phi\}$ is a vector of potentials defined at the nodes resulting from the boundary element discretization, $\{\Phi_{,n}\}$ is a vector of boundary normal derivatives, $\{\Psi\}$ is a vector of distributed domain sources, and $[G]$ and $[F]$ are boundary element system matrices. For a boundary element, e, these matrices are given as

$$[F]_e = \int_0^1 F[N]J\,\mathrm{d}\eta \tag{5}$$

$$[G]_e = \int_0^1 G[N]J\,\mathrm{d}\eta, \tag{6}$$

where $[N]$ is a matrix of shape functions employed in the boundary element discretization,

J is the transformation Jacobian and η is a nondimensional coordinate. The system of equations given in eqn (4) is now rearranged such that all unknown boundary quantities appear on the left hand side, resulting in

$$[K]\{Y\} = [H]\{Z\} + \{\Psi\} \tag{7}$$

where $\{Y\}$ is a vector of unknown boundary quantities, $\{Z\}$ is a vector of prescribed boundary conditions, and $[K]$ and $[H]$ are rearranged forms of the system matrices $[F]$ and $[G]$, respectively. The particular integral approach (Pape and Banerjee, 1987; Henry and Banerjee, 1988) is employed for the treatment of the discretized distributed sources. The vector of external sources, following this technique, may be written as

$$\{\Psi\} = [F]\{\Phi^p\} - [G]\{\Phi^p_n\}, \tag{8}$$

where $\{\Phi^p\}$ and $\{\Phi^p_n\}$ are the vectors of particular integral solutions for potentials and normal derivatives, respectively. After solving the system given in eqn (7) for the unknown boundary variables, the potentials at interior points may be calculated. The integral representation given in eqn (2) is now written for a point, ξ, inside the domain, Ω, as

$$\Phi(\xi) = \Phi^p(\xi) + \int_\Gamma [G(\mathbf{x},\xi)\Phi^c_n(\mathbf{x}) - F(\mathbf{x},\xi)\Phi^c(\mathbf{x})]\,\mathrm{d}\Gamma(\mathbf{x}) \tag{9}$$

where $\Phi^c(\mathbf{x}) = \Phi(\mathbf{x}) - \Phi^p(\mathbf{x})$, $\Phi^c_n(\mathbf{x}) = \Phi_n(\mathbf{x}) - \Phi^p_n(\mathbf{x})$, and $\Phi^p(\mathbf{x})$ and $\Phi^p_n(\mathbf{x})$ are the values of the particular integral and its normal derivative, respectively, at a location, $\mathbf{x}$, on the boundary, Γ. The same boundary element discretization as that employed for the calculation of the unknown boundary variables is applied to eqn (9) to obtain the expression for the potential at an internal point, ξ, as

$$\Phi(\xi) = \Phi^p(\xi) + \{G(\xi)\}^{\mathrm{T}}\{\Phi^c_n\} - \{F(\xi)\}^{\mathrm{T}}\{\Phi^c\}, \tag{10}$$

where $\{\Phi^c_n\} = \{\Phi_n\} - \{\Phi^p_n\}$, $\{\Phi^c\} = \{\Phi\} - \{\Phi^p\}$, and $\{G(\xi)\}$ and $\{F(\xi)\}$ are the vectors of discretized boundary element kernels, $G(\mathbf{x},\xi)$ and $F(\mathbf{x},\xi)$, respectively. These vectors are generated similarly to the rows of the boundary element system matrices $[G]$ and $[F]$. Since the point ξ does not belong to the boundary Γ, no singular integrations are encountered in these calculations.

3. STOCHASTIC BOUNDARY ELEMENT METHOD

The domain, Ω, is assumed to be characterized by a set of random parameters, b_i, $i = 1, 2, \ldots, q$, that are described as random variables with mathematical expectations, $\bar{b}_i = E[b_i]$, and covariances, $\mathrm{Cov}(b_i, b_j) = E[(b_i - \bar{b}_i)(b_j - \bar{b}_j)]$, where q is the total number of random variables. The random variables, b_i, may represent the random geometric configuration or the random material parameter of the domain. The boundary conditions and the external sources acting on the object are assumed to be deterministic. The general perturbation formulation, developed originally for the treatment of two-dimensional problems of elastostatics (Kaljević and Saigal, 1993), is extended for the treatment of potential problems with a random differential operator. The salient features of the procedure are now briefly reviewed.

The quantities that appear in eqn (4) and that depend on the random parameters describing the problem are expanded in a Taylor series about the mathematical expectations of the random parameters. Retaining up to second order terms in this expansion, the vector of unknown boundary variables is written as

$$\{Y\} = \{\bar{Y}\} + \sum_{i=1}^{q} \left\{\frac{\partial \bar{Y}}{\partial b_i}\right\} db_i + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q} \left\{\frac{\partial^2 \bar{Y}}{\partial b_i \partial b_j}\right\} db_i\, db_j, \tag{11}$$

where an overbar $(-)$ denotes the quantities that are calculated at the mathematical expectations of the random variables; $db_i = b_i - \bar{b}_i$ and $\partial/\partial b_i$ denotes the derivative with respect to a random variable, b_i. Similar expressions may be written for other quantities in eqn (4) that depend on the random parameters. The Taylor series expansions, such as in eqn (11), are introduced into eqn (4), and after neglecting higher order terms, and equating terms of the same order, the systems of equations for response variables, and their respective first and second order derivatives are obtained as

(a) order zero

$$[\bar{K}]\{\bar{Y}\} = [\bar{H}]\{Z\} + \{\bar{\Psi}\} \tag{12}$$

(b) order one

$$[\bar{K}]\left\{\frac{\partial \bar{Y}}{\partial b_i}\right\} = \left[\frac{\partial \bar{H}}{\partial b_i}\right]\{Z\} - \left[\frac{\partial \bar{K}}{\partial b_i}\right]\{\bar{Y}\} + \left\{\frac{\partial \bar{\Psi}}{\partial b_i}\right\}, \quad i = 1, 2, \ldots, q \tag{13}$$

(c) order two

$$[K]\{\bar{Y}_2\} = -\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\frac{\partial \bar{K}}{\partial b_i}\right]\left\{\frac{\partial \bar{Y}}{\partial b_j}\right\}\operatorname{Cov}(b_i, b_j) + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\frac{\partial^2 \bar{H}}{\partial b_i \partial b_j}\right]\operatorname{Cov}(b_i, b_j)\{Z\}$$
$$-\frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left[\frac{\partial^2 \bar{K}}{\partial b_i \partial b_j}\right]\operatorname{Cov}(b_i, b_j)\{\bar{Y}\} + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left\{\frac{\partial^2 \bar{\Psi}}{\partial b_i \partial b_j}\right\}\operatorname{Cov}(b_i, b_j), \tag{14}$$

where $\{\bar{Y}_2\}$ is given as

$$\{\bar{Y}_2\} = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q}\left\{\frac{\partial^2 \bar{Y}}{\partial b_i \partial b_j}\right\}\operatorname{Cov}(b_i, b_j). \tag{15}$$

Solutions of eqns (12)–(14) are used to calculate the response statistics of the unknown boundary variables as

$$E[\{Y\}] = \{\bar{Y}\} + \{\bar{Y}_2\} \tag{16}$$

$$\operatorname{Cov}(Y_i, Y_j) = \sum_{m=1}^{q}\sum_{n=1}^{q}\frac{\partial \bar{Y}_i}{\partial b_m}\frac{\partial \bar{Y}_j}{\partial b_n}\operatorname{Cov}(b_m, b_n). \tag{17}$$

Detailed expressions for the case when the random variables that describe the random parameters are uncorrelated were given by Kaljević and Saigal (1993). The general formulation defined by eqns (12)–(17) is now specialized for the cases of random geometric configuration and random material parameter, respectively.

4. RANDOM GEOMETRIC CONFIGURATION

The domain, Ω, bounded by the contour, Γ, is considered here. It is assumed that the contour Γ consists of two parts: a deterministic portion, Γ_d, and a portion, Γ_r, that is characterized by a finite set of q random parameters. The parametric equations of the contour, Γ_r, are given as

$$x_i = x_i(b_1, b_2, \ldots, b_q, t), \quad i = 1, 2, \tag{18}$$

where x_i, $i = 1, 2$, are the Cartesian coordinates of a location $\mathbf{x}$ on Γ_r; b_j, $j = 1, 2, \ldots, q$, are the random parameters that are given by their mathematical expectations, $\bar{b}_j$, and their covariances, $C(b_i, b_j)$, and t is a curve parameter. For the case of random configuration, the derivatives of the boundary element matrix $[F]_e$ are calculated as

$$[F]_{e,l} = \int_0^1 (F_{,l}J + FJ_{,l})[N]\,\mathrm{d}\eta \tag{19a}$$

$$[F]_{e,lm} = \int_0^1 (F_{,lm}J + F_{,l}J_{,m} + F_{,m}J_{,l} + FJ_{,lm})[N]\,\mathrm{d}\eta, \tag{19b}$$

where a comma denotes differentiation with respect to a random variable, and the subscripts l and m denote the random variables b_l and b_m, respectively. Similar expressions can be written for the derivatives of the element matrix $[G]_e$. The derivative matrices that appear in eqns (13) and (14) may be generated using standard assembly procedures for boundary element matrices. The derivatives of the particular integrals are given as

$$\{\Psi\}_{,l} = [F]_{,l}\{\Phi^p\} + [F]\{\Phi^p\}_{,l} - ([G]_{,l}\{\Phi^p_n\} + [G]\{\Phi^p_n\}_{,l}) \tag{20a}$$

$$\{\Psi\}_{,lm} = [F]_{,lm}\{\Phi^p\} + [F]_{,m}\{\Phi^p\}_{,l} + [F]_{,l}\{\Phi^p\}_{,m} + [F]\{\Phi^p\}_{,lm} - ([G]_{,lm}\{\Phi^p_n\} + [G]_{,m}\{\Phi^p_n\}_{,l} + [G]_{,l}\{\Phi^p_n\}_{,m} + [G]\{\Phi^p_n\}_{,lm}). \tag{20b}$$

Explicit expressions for the derivatives of the boundary element kernels G and F with respect to the random variables, as well as the derivatives of the particular integrals for three different cases of distributed sources, are given in the Appendix. Equations (A1)–(A10) are valid for an arbitrary random contour that may be described by a set of correlated random variables. Only the derivatives of the coordinates with respect to the random variables are dependent on the random curve. For the cases of circular and elliptical contours, the derivatives of the coordinates x and y are easily derived in closed form. The parametric equations for these contours are given as

(a) circular contour

$$x = x_o + R\cos\vartheta, \quad y = y_o + R\sin\vartheta, \tag{21}$$

where (x_o, y_o, R) are the geometry parameters. The coordinate pair (x_o, y_o) defines the location of the center of the circular arc, and R is the radius.

(b) elliptical contour

$$x = x_s + a\cos\vartheta\cos\varphi - b\sin\vartheta\sin\varphi, \quad y = y_s + a\cos\vartheta\sin\varphi + b\sin\vartheta\cos\varphi, \tag{22}$$

where $(x_s, y_s, a, b, \varphi)$ are the geometry parameters. (x_s, y_s) are the coordinates of the center of the elliptical arc, (a, b) are the semiaxes, and φ is the inclination of the ellipse semiaxes. The availability of the derivatives of these contours in the closed form enables all derivatives of the boundary element matrices required in the present analysis to be evaluated analytically. The derivatives of the boundary element kernels with respect to the random parameters involve the derivatives with respect to coordinates which may result in higher order singularities of the derivative kernels. Special care must therefore be exercised in these calculations. The strongly singular terms of the matrix $[F]$ and its derivatives may not be calculated directly. An indirect procedure that utilizes a state of uniform unit potentials prescribed along the contour of the object along with zero distributed sources (Banerjee and Butterfield, 1981) is used to calculate the diagonal terms of the matrix $[F]$. An implicit differentiation scheme (Kaljević and Saigal, 1993; Saigal *et al.*, 1989) may then be employed to calculate the diagonal terms of the first and second order derivatives of the matrix $[F]$. The singular terms of the derivative kernel, G, however, are calculated directly. It is noted that the singular term in the kernel G for potential problems has the same form as that for two-dimensional elastostatics. It was shown by Kaljević and Saigal (1993) that the derivatives of the elastostatics kernel G do not involve higher order singularities. This permits the use of standard integration schemes for the calculation of the derivatives of the matrix $[G]$.

The solutions for the boundary variables and their derivatives obtained from eqns (12)–(14) are now used to calculate the response statistics of internal potentials. The first and second order derivatives of the internal potentials with respect to the random variables are given as

$$\Phi_{,l}(\xi) = \Phi^{p}_{,l}(\xi) + \{G(\xi)\}^{\mathrm{T}}_{,l}\{\Phi^{c}_{n}\} + \{G(\xi)\}^{\mathrm{T}}\{\Phi^{c}_{n}\}_{,l} - (\{F(\xi)\}^{\mathrm{T}}_{,l}\{\Phi^{c}\} + \{F(\xi)\}^{\mathrm{T}}\{\Phi^{c}\}_{,l}) \tag{23a}$$

$$\begin{aligned}\Phi_{,lm}(\xi) = {} & \Phi^{p}_{,lm}(\xi) + \{G(\xi)\}^{\mathrm{T}}_{,lm}\{\Phi^{c}_{n}\} + \{G(\xi)\}^{\mathrm{T}}_{,l}\{\Phi^{c}_{n}\}_{,m} + \{G(\xi)\}^{\mathrm{T}}_{,m}\{\Phi^{c}_{n}\}_{,l} + \{G(\xi)\}^{\mathrm{T}}\{\Phi^{c}_{n}\}_{,lm} \\ & - (\{F(\xi)\}^{\mathrm{T}}_{,lm}\{\Phi^{c}\} + \{F(\xi)\}^{\mathrm{T}}_{,l}\{\Phi^{c}\}_{,m} + \{F(\xi)\}^{\mathrm{T}}_{,m}\{\Phi^{c}\}_{,l} + \{F(\xi)\}^{\mathrm{T}}\{\Phi^{c}\}_{,lm}).\end{aligned} \tag{23b}$$

These derivatives are introduced into eqns (16) and (17) to obtain the mathematical expectation and the variance of the internal potential at a location ξ. It is noted that the derivatives of the internal potentials involve the calculation of the derivatives of the coordinates with respect to the random variables. The modeling of the interior of the domain is therefore required in order to calculate the response statistics of internal potentials. Several models to characterize the randomness of the interior of the object are presented in the section on numerical examples. The influence of various assumptions regarding the randomness of the internal points on response statistics is studied. It is seen from the numerical data obtained that these assumptions do not affect the response statistics of the boundary variables.

5. RANDOM MATERIAL PARAMETER

The domains whose material properties are characterized by a random material parameter are analysed. It is seen from eqn (3) that the boundary element kernels $G(\mathbf{x}, \xi)$ and $F(\mathbf{x}, \xi)$ do not depend on the random material parameter, k, and that in the absence of distributed sources, the potential flow through a homogeneous domain also does not depend on the material parameter. It may be concluded that for such cases, the random material parameter does not affect the response of the object. Random effects related to material characteristics are introduced only through the action of external sources. Such effects are present even when the sources are described by deterministic functions. It is assumed that the contribution of external sources to the response may be expressed in terms of a particular integral term, as given in eqn (8). The random parameter, k, is modeled as a homogeneous random field with a constant mathematical expectation, $\bar{k}$, and a covariance function, $\mathrm{Cov}(\mathbf{x}, \mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are two points within the domain, Ω. After discretizing the random field that describes the material parameter, k, into a set of, in general, correlated random variables, the equations for the response variables and their derivatives are obtained as

(a) order zero:

$$[K]\{\bar{Y}\} = [H]\{\bar{Z}\} + \{\bar{\Psi}\} \tag{24}$$

(b) order one:

$$[K]\left\{\frac{\partial \bar{Y}}{\partial b_i}\right\} = \left\{\frac{\partial \bar{\Psi}}{\partial b_i}\right\} \quad i = 1,2,\ldots,s \tag{25}$$

(c) order two:

$$[K]\{\bar{Y}_2\} = \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s}\left\{\frac{\partial^2 \bar{\Psi}}{\partial b_i \partial b_j}\right\}\operatorname{Cov}(b_i, b_j) \tag{26}$$

$$\{\bar{\Psi}\}_{,l} = [F]\{\bar{\Phi}^p\}_{,l} - [G]\{\bar{\Phi}^p_n\}_{,l} \tag{27a}$$

$$\{\bar{\Psi}\}_{,lm} = [F]\{\bar{\Phi}^p\}_{,lm} - [G]\{\bar{\Phi}^p_n\}_{,lm}, \tag{27b}$$

where s is the number of correlated random variables, and $\{\bar{\Phi}^p\}_{,l}$ and $\{\bar{\Phi}^p_n\}_{,l}$ represent the derivatives of the particular integral and its normal derivative with respect to the random variables calculated at their mathematical expectations.

It is seen from eqns (A5), (A7) and (A9) given in the Appendix that the particular integral term may be written as

$$\Phi^p = -\frac{1}{k}\chi(x_1, x_2), \tag{28}$$

where $\chi(x_1, x_2)$ is a function of the spatial coordinates x_1 and x_2 and depends on the form of the variation of the distributed source, Ψ, across the domain. A similar expression may also be written for the particular integral Φ^p_n. For the special case of random field discretization given as

$$k(\mathbf{x}) = \sum_{m=1}^{s} b_m M_m(\mathbf{x}), \tag{29}$$

where $M_m(\mathbf{x})$ are the interpolation functions defined as

$$M_m(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Gamma_m \\ 0, & \text{otherwise} \end{cases} \tag{30}$$

and Γ_m denotes the domain of the boundary element m, the derivatives of the particular integral Φ^p are given as

$$\frac{\partial \bar{\Phi}^p}{\partial b_m} = \frac{1}{\bar{k}_m^2}\chi(x_1, x_2) \tag{31a}$$

$$\frac{\partial^2 \bar{\Phi}^p}{\partial b_m^2} = -\frac{2}{\bar{k}_m^3}\chi(x_1, x_2). \tag{31b}$$

The mixed second order derivatives for this case are all equal to zero.

A transformation of the correlated random variables into an uncorrelated set (Liu *et al.* 1986b; Kaljević and Saigal, 1993) is performed to reduce the number of matrix operations. For an uncorrelated set of random variables, c_i, $i = 1,2,\ldots,s$, defined by transformation

$$\{c\} = [\Theta]^{\mathrm{T}}\{b\}, \tag{32}$$

where $\{b\}$ is the vector of original random variables resulting from the random field discretization, $\{c\}$ is the vector of transformed uncorrelated random variables, and $[\Theta]$ is the matrix of eigenvectors of the discretized covariance matrix $[D]$ whose terms are given as $d_{ij} = \mathrm{Cov}(b_i, b_j)$, the system of equations for the terms of order two is given as

$$[K]\{\bar{Y}_2\} = \frac{1}{2}\sum_{i=1}^{s}\left\{\frac{\partial^2\bar{\Psi}}{\partial c_i^2}\right\}\mathrm{Var}\,(c_i) \tag{33}$$

$$\{\bar{Y}_2\} = \frac{1}{2}\sum_{i=1}^{s}\left\{\frac{\partial^2\bar{Y}}{\partial c_i^2}\right\}\mathrm{Var}\,(c_i), \tag{34}$$

where $\mathrm{Var}\,(c_i)$ denotes the variance of the random variable c_i. The systems of equations for the respective orders zero and one have the same form as those given in terms of the original random variables, $\{b\}$. The derivatives of a scalar function, f, with respect to the original random variables are expressed in terms of the derivatives with respect to the transformed variables as

$$\frac{\partial f}{\partial b_i} = \sum_{l=1}^{s}\frac{\partial f}{\partial c_l}\theta_{il} \tag{35a}$$

$$\frac{\partial^2 f}{\partial b_i \partial b_j} = \sum_{l=1}^{s}\sum_{m=1}^{s}\frac{\partial^2 f}{\partial c_l \partial c_m}\theta_{il}\theta_{jm}. \tag{35b}$$

The derivatives of the vector functions are obtained similarly.

The expressions for the response statistics of the internal potentials are derived next. The derivatives of the potential, Φ, at an internal point, ξ, with respect to a transformed random variable, c_i, are given as

$$\bar{\Phi}_{,c_i}(\xi) = \bar{\Phi}^p_{,c_i}(\xi) + \{G(\xi)\}^{\mathrm{T}}\{\bar{\Phi}^c_n\}_{,c_i} - \{F(\xi)\}^{\mathrm{T}}\{\bar{\Phi}\}_{,c_i} \tag{36a}$$

$$\bar{\Phi}_{,c_ic_i}(\xi) = \bar{\Phi}^p_{,c_ic_i}(\xi) + \{G(\xi)\}^{\mathrm{T}}\{\bar{\Phi}^c_n\}_{,c_ic_i} - \{F(\xi)\}^{\mathrm{T}}\{\bar{\Phi}\}_{,c_ic_i}, \tag{36b}$$

where the transformed derivatives of the particular integral terms at a point, ξ, are calculated using eqns (35a,b). It is seen from eqns (36a,b) that the discretization of the random field defined by eqns (29) and (30) is not sufficient for the calculation of the response statistics of the internal potentials. Additional random variables need to be defined inside the domain in order to calculate the derivatives of the particular integrals at the internal points. A modified discretization of the random field may be performed by retaining the random variables defined on the contour as depicted in eqns (29) and (30), and introducing p additional random variables, one for each internal point for which the response statistics are required. It is assumed that these additional random variables are defined on the subdomains Ω_j, $j = 1, 2, \ldots, p$, where each subdomain contains only one internal point ξ_j and $\cup\Omega_j = \Omega$. The discretization of the random field is now given as

$$k(\mathbf{x}) = \sum_{m=1}^{s} b_m M_m(\mathbf{x}) + \sum_{m=s+1}^{s+p} b_m N_m(\mathbf{x}), \tag{37}$$

where the domain interpolation functions $N_m(\mathbf{x})$ are defined as

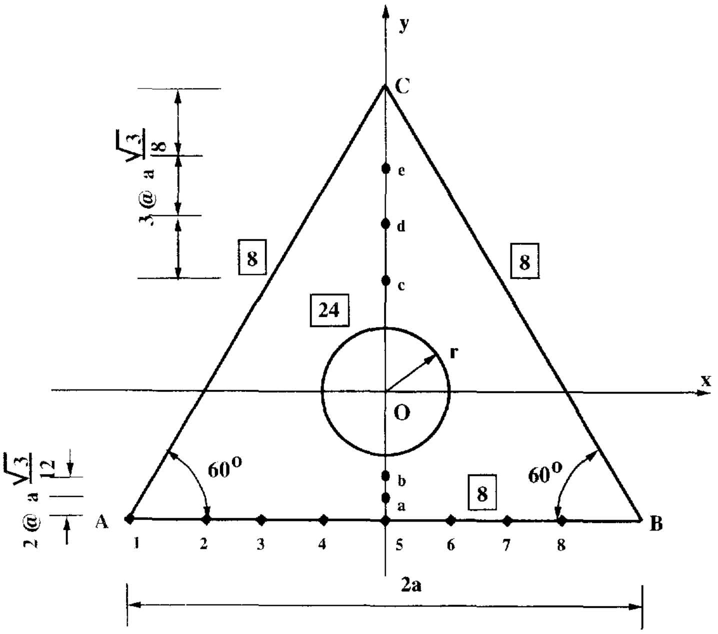

Fig. 1. A triangular plate with a circular hole.

$$N_m(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_m \\ 0, & \text{otherwise} \end{cases}. \tag{38}$$

For such a discretization the systems of equations for the unknown variables and their derivatives given in eqns (24)–(26) remain unchanged.

6. NUMERICAL EXAMPLES

The above formulations are applied to a number of problems of potential flow through domains characterized by random parameters. The domains with a random geometric configuration are studied first. It is assumed that the contour of the domain consists of deterministic portions of arbitrary shape, and of random portions of circular and elliptical shapes, respectively. The random circular and elliptical contours are modeled using discrete sets of random variables. Uncorrelated variables are employed in the present calculations without a loss of generality, since the correlated random variables may be transformed into an uncorrelated set using eqn (32). The domains characterized by a random material parameter are analysed next. The random material parameter is modeled as a homogeneous random field that is described by a constant mathematical expectation and an exponential covariance.

Boundary element discretizations employed in the numerical examples are obtained from convergence studies performed for corresponding deterministic problems. The response statistics obtained by the present formulations are compared with the results of Monte Carlo simulations performed using 5000 samples. The random variables for Monte Carlo simulations are generated using the procedures outlined by Press *et al.* (1990). Numerical values used to describe the geometric configuration, the material parameter and the distributed sources are given without dimensions to make the present results applicable to a variety of physical problems governed by Poisson's equation.

6.1. *A triangular plate with a circular hole*

The potential flow through a triangular plate with a circular hole, as shown in Fig. 1, is analysed. The outer contour of the domain is of the shape of an equilateral triangle with side length $2a$ and is modeled as deterministic. The inner, circular contour is assumed to be

Table 1. Response statistics of surface potentials for a triangular plate with a circular hole along line AB

Loc.	Deterministic solution	This study		Monte Carlo	
		$E[\Phi]$	Var $[\Phi]$	$E[\Phi]$	Var $[\Phi]$
1	0.9887	0.9877	0.3285×10^{-4}	0.9874	0.4015×10^{-4}
2	0.9884	0.9875	0.3401×10^{-4}	0.9872	0.4144×10^{-4}
3	0.9863	0.9855	0.4438×10^{-4}	0.9850	0.4669×10^{-4}
4	0.9761	0.9753	0.9963×10^{-4}	0.9748	0.1094×10^{-3}
5	0.9322	0.9315	0.3530×10^{-3}	0.9312	0.3533×10^{-3}
6	0.7959	0.7962	0.6970×10^{-3}	0.7961	0.6906×10^{-3}
7	0.5557	0.5564	0.4280×10^{-3}	0.5565	0.4299×10^{-3}
8	0.2737	0.2740	0.6760×10^{-4}	0.2740	0.6830×10^{-4}

random and is described by three random variables: the radius, r, and the coordinates of the center, x_0 and y_0. These variables are characterized by their respective mathematical expectations, $\bar{r}$, $\bar{x}_0$ and $\bar{y}_0$, and standard deviations, α_r, α_{x0} and α_{y0}. The mathematical expectation, $(\bar{x}_0, \bar{y}_0)$, of the random circular contour coincides with the centroid of the outer contour. The numerical values for the geometric and material parameters of the domain are chosen as: side length, $2a = 12$ units, and the material parameter, $k = 1$ units. The deterministic boundary conditions for the potential flow are given as: $\Phi_n = 0$ along the sides AB and AC of the outer contour; linearly varying potential along the side BC from $\Phi(B) = 0$ to $\Phi(C) = 1.0$ units; and constant potential $\Phi = 1$ unit along the circular contour. No distributed sources are present. The contour of the domain is discretized using 48 conforming quadratic boundary elements, as shown in Fig. 1. The numbers in square boxes in Fig. 1 denote the number of boundary elements used to discretize the corresponding portion of the contour.

The analysis is first performed for the case when all three parameters that describe the circular contour are assumed to be random. The statistics of random variables that describe these parameters are taken as: $\bar{r} = 1.732$ units, $\bar{x}_0 = 0.0$, $\bar{y}_0 = 0.0$, $\alpha_r = 0.1732$ units, $\alpha_{x0} = 0.12$ units and $\alpha_{y0} = 0.12$ units. The response statistics for both surface potentials and internal potentials are computed. The results for the mathematical expectations and variances of potentials at locations 1–8 placed at equal distances along the side AB of the outer contour, together with the solution of the corresponding deterministic problem, and the results of the Monte Carlo simulation using 5000 samples, are given in Table 1. A good agreement is observed between the results obtained from the present formulations and those from Monte Carlo simulations, except for the variances at points 1 and 2. These larger discrepancies may be attributed to the corner positions of these locations (Banerjee and Butterfield, 1981) where difficulties in the calculation of the derivatives of response variables are more pronounced than on the smooth portions of the contour.

The response statistics for the internal potentials are calculated corresponding to two distinct assumptions on the randomness of internal points: (a) the randomness of the circular contour does not affect the interior of the domain (the positions of the internal points are assumed to be deterministic), and (b) a portion of the interior enclosed by the circle with the center $(\bar{x}_0, \bar{y}_0)$ and radius $2\bar{r}$ is assumed to be random, while the rest of the domain is deterministic. For the deterministic portion of the domain, the derivatives of the coordinates with respect to the random parameters are all equal to zero. The random portion of the domain for case (b) is modeled as

$$x = X + R\cos\varphi \tag{39a}$$

$$y = Y + R\sin\varphi, \tag{39b}$$

where x and y are the Cartesian coordinates of the internal point, $X = x_0(1-t)+\bar{x}_0 t$, $Y = y_0(1-t)+\bar{y}_0 t$, $R = r+(2\bar{r}-r)t$, and $0 \leqslant t \leqslant 1$. The nonvanishing derivatives of X, Y and R, with respect to the random parameters, are $X_{,x0} = 1-t$, $Y_{,y0} = 1-t$, and $R_{,r} = 1-t$. All other first order derivatives and all second order derivatives are equal to zero.

Table 2. Response statistics of internal potentials for a triangular plate with a circular hole along the Y-axis

Loc.	Deterministic solution	Mathematical expectations: This study $r_d = \bar{r}$	Mathematical expectations: This study $r_d = 2\bar{r}$	Mathematical expectations: Monte Carlo simulation	Variances ($\times 10^{-3}$): This study $r_d = \bar{r}$	Variances ($\times 10^{-3}$): This study $r_d = 2\bar{r}$	Variances ($\times 10^{-3}$): Monte Carlo simulation
a	0.9362	0.9355	0.9379	0.9351	0.3540	0.3090	0.3535
b	0.9485	0.9476	0.9502	0.9473	0.3530	0.2090	0.3502
c	0.8917	0.8920	0.8953	0.8920	0.1920	0.1270	0.1928
d	0.8634	0.8635	0.8665	0.8635	0.0386	0.0386	0.0388
e	0.8955	0.8956	0.8955	0.8956	0.0040	0.0040	0.0040

The mathematical expectations and variances of the internal potentials for locations $a-e$ along the Y-axis, for the two assumptions on the randomness of the interior, are given in Table 2. The solutions for the corresponding deterministic problem and the results of Monte Carlo simulations using 5000 samples are also given in Table 2 for the assumption of a deterministic interior. Again, a good agreement between the results of the present formulations and those of the Monte Carlo simulation is observed. It is seen from Table 2 that the response statistics of the internal potentials are affected by the assumption on the randomness of interior points. The response statistics for points d and e remain the same for both cases, since they belong to the portion that is assumed to be deterministic in both cases.

In order to assess the influence of the random parameters describing the circular contour on the response statistics, the computations are repeated for the cases when each of the parameters individually is assumed to be random, while the other two are kept deterministic. The response statistics of the internal potentials at locations a–e, obtained by the present formulation using the assumption of a deterministic interior, are given in Table 3. It is seen from the results in Table 3 that the randomness of the variable, y_0, does not affect the response statistics, and that the most significant effect is caused by the variation of the radius of the circular contour.

6.2. *A circular annulus under distributed sources*

The flow through a circular annulus with a deterministic outer contour of radius, r_b, and a random inner contour is analysed. The inner circular contour is described using three random variables, r_a, x_0 and y_0, as in Section 6.1. The numerical data are assumed as: $r_b = 1.0$ units and $k = 1.0$ units. The flow is subjected to deterministic boundary conditions given as: $\Phi_n = 0$ along the inner contour and $\Phi = 2\alpha/\pi$ along the outer contour, where the angle, α, is measured counterclockwise from the positive x-axis. Each contour of the domain of the object is discretized using 16 quadratic boundary elements.

The analysis is performed to first verify the present formulations for the flow due to distributed sources. Three distinct sources are considered: (a) $\Psi(x, y) = \Psi_0$; (*b*) $\Psi(x, y) = \Psi_0 xy/r_b^2$; and (c) $\Psi(x, y) = \Psi_0(x^2+y^2)/r_b^2$, where x and y are coordinates of an arbitrary point in the interior of the domain, and Ψ_0 is the reference intensity of the

Table 3. Influence of random geometry parameters on the response statistics for a triangular plate with a circular hole

Loc.	Radius R: $E[\Phi]$	Radius R: Var $[\Phi]$	Coordinate x_0: $E(\Phi]$	Coordinate x_0: Var $[\Phi]$	Coordinate y_0: $E[\Phi]$	Coordinate y_0: Var $[\Phi]$
a	0.9357	0.2622×10^{-3}	0.9361	0.7502×10^{-4}	0.9360	0.1720×10^{-4}
b	0.9478	0.2591×10^{-3}	0.9485	0.8030×10^{-4}	0.9483	0.1354×10^{-4}
c	0.8920	0.1391×10^{-3}	0.8919	0.5125×10^{-4}	0.8916	0.1899×10^{-5}
d	0.8635	0.2835×10^{-4}	0.8635	0.9994×10^{-5}	0.8634	0.2549×10^{-6}
e	0.8956	0.2945×10^{-5}	0.8956	0.1035×10^{-5}	0.8955	0.1525×10^{-7}

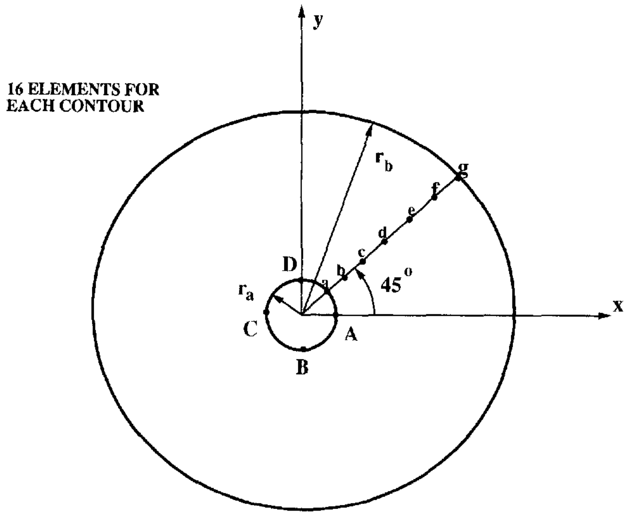

Fig. 2. A circular annulus under distributed sources.

source. The distributed sources are treated using particular integrals. The particular integrals, and their respective first and second order derivatives with respect to the random variables, for these three cases of distributed sources are given in the Appendix. For each case the reference intensity is assumed as $\Psi_0 = 1.0$ units. The statistics of the random inner contour parameters are assumed as: $\bar{r}_a = 0.3$ units, $\bar{x}_0 = 0.0$, $\bar{y}_0 = 0.0$, $\alpha_r = 0.03$ units, $\alpha_{x0} = 0.12$ units and $\alpha_{y0} = 0.12$ units. The response statistics for potentials at locations A, B, C and D of the annulus, as shown in Fig. 2, for the three cases of distributed sources considered here are given in Table 4. The results obtained using the present formulations, together with the corresponding deterministic solution, and the results of the Monte Carlo simulation using 5000 samples are given in Table 4. A good agreement of the results is observed verifying the developments for the treatment of distributed sources.

The response statistics of internal potentials at locations b–f shown in Fig. 2 are calculated for the distributed source of a constant intensity, $\Psi = 1.0$ units. The points b–f are placed at equal distances along line ag. The positions of the internal points are described by their Cartesian coordinates given in eqns (39a,b). X, Y and R are now defined as

$$R = \begin{cases} r_a + \dfrac{r_b}{r_d}(r_d - r_a)t & 0 \leqslant t \leqslant \dfrac{r_d}{r_b} \\ tr_b & \dfrac{r_d}{r_b} \leqslant t \leqslant 1 \end{cases} \tag{40a}$$

$$X = \begin{cases} x_0\left(1 - \dfrac{r_b}{r_d}t\right) + \bar{x}_0\dfrac{r_b}{r_d}t & 0 \leqslant t \leqslant \dfrac{r_d}{r_b} \\ \bar{x}_0 & \dfrac{r_d}{r_b} \leqslant t \leqslant 1 \end{cases} \tag{40b}$$

$$Y = \begin{cases} y_0\left(1 - \dfrac{r_b}{r_d}t\right) + \bar{y}_0\dfrac{r_b}{r_d}t & 0 \leqslant t \leqslant \dfrac{r_d}{r_b}, \\ \bar{y}_0 & \dfrac{r_d}{r_b} \leqslant t \leqslant 1 \end{cases} \tag{40c}$$

where $r_a \leqslant r_d \leqslant r_b$. It is noted from eqns (40a–c) that a portion of the domain enclosed by

Table 4. Response statistics of surface potentials for a circular annulus under various distributed sources

Loc.		$\Psi = \Psi_0$ This study	$\Psi = \Psi_0$ Monte Carlo	$\Psi = \Psi_0 \cdot xy/R^2$ This study	$\Psi = \Psi_0 \cdot xy/R^2$ Monte Carlo	$\Psi = \Psi_0 \cdot (x^2+y^2)/R^2$ This study	$\Psi = \Psi_0 \cdot (x^2+y^2)/R^2$ Monte Carlo
A	Determ.	0.1733	—	0.0000	—	0.0596	—
	$E[\Phi]$	0.1732	0.1731	0.0000	0.0000	0.0594	0.0594
	Var $[\Phi]$	0.252×10^{-3}	0.251×10^{-3}	0.135×10^{-3}	0.137×10^{-3}	0.126×10^{-3}	0.128×10^{-3}
B	Determ.	-0.3772	—	-0.5505	—	-0.4909	—
	$E[\Phi]$	-0.3761	-0.3761	-0.5492	-0.5493	-0.4898	-0.4898
	Var $[\Phi]$	0.342×10^{-2}	0.338×10^{-2}	0.223×10^{-2}	0.221×10^{-2}	0.233×10^{-2}	0.230×10^{-2}
C	Determ.	0.1733	—	0.0000	—	0.0596	—
	$E[\Phi]$	0.1732	0.1732	0.0000	0.0000	0.0594	0.0594
	Var $[\Phi]$	0.252×10^{-3}	0.252×10^{-3}	0.116×10^{-3}	0.118×10^{-3}	0.126×10^{-3}	0.128×10^{-3}
D	Determ.	0.7238	—	0.5505	—	0.6100	—
	$E[\Phi]$	0.7224	0.7224	0.5492	0.5492	0.6087	0.6087
	Var $[\Phi]$	0.129×10^{-2}	0.223×10^{-2}	0.126×10^{-2}	0.220×10^{-2}	0.213×10^{-2}	0.210×10^{-2}

a concentric circle with radius r_d is random, whereas the rest of the domain is deterministic. The derivatives of the coordinates x and y with respect to the random parameter, b, are given as

$$x_{,b} = R_{,b}\cos\varphi + X_{,b} \tag{41a}$$

$$y_{,b} = R_{,b}\sin\varphi + Y_{,b}, \tag{41b}$$

where b may represent any of the three parameters that describe the random circular contour. Similar expressions may be written for the second order derivatives of the coordinates x and y. The nonvanishing derivatives of X, Y and R, with respect to the random variables, are given as

$$R_{,r_a} = X_{,x_0} = Y_{,y_0} = \begin{cases} 1 - \dfrac{r_b}{r_d}t & 0 \leqslant t \leqslant \dfrac{r_d}{r_b} \\ 0 & \dfrac{r_d}{r_b} \leqslant t \leqslant 1 \end{cases}. \tag{42}$$

The remaining first order derivatives and all second order derivatives are equal to zero. The response statistics are calculated for three different values of r_d: (a) $r_d = \bar{r}_a$; (b) $r_d = 0.65r_b$; and (c) $r_d = r_b$. Case (a) corresponds to the deterministic interior, case (b) represents a partially random domain and case (c) assumes the entire domain to be random. The response statistics of internal potentials at locations b–f for all three cases of random interior, together with the solution of the corresponding deterministic problem, and the results of Monte Carlo simulation using 5000 samples for the case when $r_d = \bar{r}_a$ are given in Table 5. A good agreement of the results obtained by the present formulation and from

Table 5. Influence of the randomness of internal points on the response statistics of internal potentials of a circular annulus

Loc.	Deterministic solution	Mathematical expectations: This study $r_d = \bar{r}_a$	This study $r_d = 0.65r_b$	This study $r_d = r_b$	Monte Carlo simulation	Variances ($\times 10^{-4}$): This study $r_d = \bar{r}_a$	This study $r_d = 0.65r_b$	This study $r_d = r_b$	Monte Carlo simulation
b	0.5776	0.5779	0.5770	0.5769	0.5780	1.9750	3.3920	4.0060	1.9290
c	0.6061	0.6063	0.6060	0.6056	0.6063	0.8185	1.3360	2.2440	0.7983
d	0.6365	0.6366	0.6366	0.6363	0.6366	0.3387	0.3387	1.0490	0.3305
e	0.6647	0.6647	0.6647	0.6645	0.6647	0.1200	0.1200	0.3778	0.1169
f	0.6886	0.6886	0.6886	0.6885	0.6886	0.0253	0.0253	0.0750	0.0246

Table 6. Comparison of probabilistic boundary element analysis and Monte Carlo simulations for variations in the standard deviation of a geometry parameter of a circular annulus

Loc.			Coefficient of variation $\delta_r = \alpha_r/\bar{r}$			
			5%	10%	15%	20%
A	$E[\Phi]$	PBE	0.1733	0.1733	0.1731	0.1730
		MC	0.1733	0.1733	0.1733	0.1729
	Var $[\Phi]$	PBE	0.294×10^{-4}	0.252×10^{-3}	0.264×10^{-3}	0.476×10^{-3}
		MC	0.289×10^{-4}	0.251×10^{-4}	0.256×10^{-4}	0.449×10^{-3}
B	$E[\Phi]$	PBE	-0.3769	-0.3761	-0.3746	-0.3727
		MC	-0.3769	-0.3761	-0.3747	-0.3728
	Var $[\Phi]$	PBE	0.806×10^{-3}	0.342×10^{-2}	0.726×10^{-2}	0.129×10^{-1}
		MC	0.796×10^{-2}	0.338×10^{-2}	0.714×10^{-2}	0.126×10^{-1}
C	$E[\Phi]$	PBE	0.1733	0.1733	0.1731	0.1731
		MC	0.1733	0.1733	0.1731	0.1729
	Var $[\Phi]$	PBE	0.294×10^{-4}	0.252×10^{-3}	0.264×10^{-3}	0.470×10^{-3}
		MC	0.289×10^{-4}	0.252×10^{-3}	0.256×10^{-3}	0.449×10^{-3}
D	$E[\Phi]$	PBE	0.7234	0.7224	0.7209	0.7186
		MC	0.7235	0.7224	0.7209	0.7187
	Var $[\Phi]$	PBE	0.308×10^{-3}	0.129×10^{-2}	0.278×10^{-2}	0.493×10^{-2}
		MC	0.305×10^{-3}	0.129×10^{-2}	0.276×10^{-2}	0.494×10^{-2}

PBE, probabilistic boundary elements; MC, Monte Carlo simulations.

the Monte Carlo simulations is observed. The Monte Carlo simulations for other two cases were not performed. It is seen from Table 5 that different assumptions on the randomness of the internal points may significantly influence the response statistics at internal points. Care must be exercised in properly assessing the randomness of the interior in order to obtain reliable results for a given analysis. It is noted that these different assumptions do not affect the response statistics on the surface of the domain.

The analysis for the distributed source of constant intensity is now performed by considering only the radius r_a to be random, while the coordinates x_0 and y_0 are considered to be deterministic. The standard deviation of the radius r_a was varied from $\alpha_{r_a} = 0.015$ units to $\alpha_{r_a} = 0.060$ units, which corresponds to a change in the coefficient of variation from $\delta_{r_a} = 5\%$ to $\delta_{r_a} = 20\%$, respectively. These analyses are performed in order to assess the range of applicability of the present formulations. The results using both the present formulation and Monte Carlo simulations are given in Table 6. It is seen from Table 6 that for this example, the present formulation provides satisfactory results for a value of the coefficient of variation of up to 20%.

6.3. *A rectangular plate with an elliptical hole*

The potential flow through a rectangular plate with an elliptical hole, as shown in Fig. 3, is analysed. The outer rectangular contour of the domain is assumed to be deterministic, and the inner elliptical contour is assumed to be random. The elliptical contour is described by five random parameters: semiaxes, a and b; coordinates of the center, x_0 and y_0; and the angle of inclination, φ. This analysis is performed in order to verify the present formulations for a different type of random contour and for a larger number of random variables. The dimensions of the outer contour are: $2p = 16$ units, $2q = 12$ units, and the material parameter is $k = 1.0$ units. The deterministic boundary conditions are prescribed as: $\Phi_n = 0$ on line AB; $\Phi_n = 1.0$ units on line CD; $\Phi = 0$ on line AD; $\Phi = 1.0$ units on line BC; and $\Phi_n = 0$ on the random elliptical contour. The contour of the domain is discretized using 68 boundary elements, as shown in Fig. 3, where the numbers in the square boxes denote the number of boundary elements used in discretizing the corresponding section of the contour. No distributed sources are present. All five parameters that describe the elliptical contour are assumed to be random, and their statistics are taken as: $\bar{a} = 2.0$ units, $\bar{b} = 1.0$ units, $\bar{x}_0 = 0.0$, $\bar{y}_0 = 0.0$, $\bar{\varphi} = 0.0$, $\alpha_a = 0.24$ units, $\alpha_b = 0.12$ units, $\alpha_{x0} = 0.10$ units,

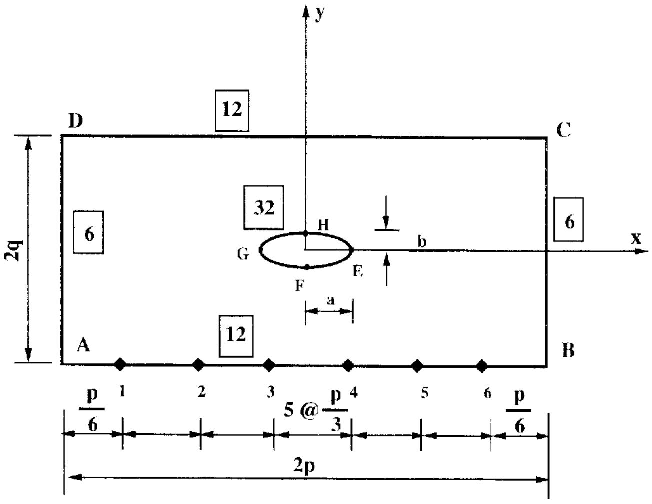

Fig. 3. A rectangular plate with an elliptical hole.

$\alpha_{y0} = 0.10$ units and $\alpha_\varphi = 0.05$ rad. The response statistics of contour potentials for the locations 1–6 on the line AB as well as for the points E, F, G and H on the elliptical contour, together with the solution of the corresponding deterministic problem, and the results of the Monte Carlo simulation using 5000 samples are given in Table 7. A good agreement of the results is observed which shows a good performance of the present formulation for a larger number of random variables employed to describe the random contour.

6.4. *A plate geometry with spatially random material parameter*

The potential flow through a domain characterized by a spatially random material parameter is analysed by considering the plate geometry shown in Fig. 4. The deterministic contour of the plate consists of the straight line segments AB, DE and EA, each of length $2a$, and the semicircle BCD with radius a, where $a = 4.0$ units. The potential flow due to a distributed source of constant intensity, $\Psi = 1.0$ unit, is analysed. The deterministic boundary conditions are prescribed as: $\Phi_{,n} = 0$ along line AE; $\Phi = 1.0$ unit along line DE; $\Phi = -2\alpha/\pi$ units, along the circular arc, BD, where the angle, α, is measured as shown in Fig. 4; and $\Phi = -1$ unit along line DE. The random material parameter, k, is modeled as a homogeneous random field, with a constant mathematical expectation, $\bar{k}$, and an exponential covariance as

$$E[k(\mathbf{x})] = \bar{k} \tag{43}$$

$$\mathrm{Cov}(k(\mathbf{x}), k(\mathbf{y})) = E[(k(\mathbf{x})-\bar{k})(k(\mathbf{y})-\bar{k})] = \sigma^2 \exp\left[-\left(\frac{|x_1-y_1|}{c_1}+\frac{|x_2-y_2|}{c_2}\right)\right] \tag{44}$$

Table 7. Response statistics for surface potentials on a rectangular plate with an elliptical hole

Loc.	Deterministic solution	This study		Monte Carlo	
		$E[\Phi]$	Var $[\Phi]$	$E[\Phi]$	Var $[\Phi]$
1	0.0776	0.0776	0.1093×10^{-5}	0.0776	0.1116×10^{-5}
2	0.2353	0.2352	0.7748×10^{-5}	0.2352	0.8012×10^{-5}
3	0.4065	0.4065	0.9253×10^{-5}	0.4065	0.9084×10^{-5}
4	0.5935	0.5935	0.8253×10^{-5}	0.5935	0.8753×10^{-5}
5	0.7647	0.7648	0.7748×10^{-5}	0.7648	0.7811×10^{-5}
6	0.9224	0.9224	0.1093×10^{-5}	0.9224	0.1099×10^{-5}
E	0.6873	0.6870	0.3217×10^{-3}	0.6874	0.3282×10^{-3}
F	0.5000	0.5000	0.1438×10^{-3}	0.5002	0.1461×10^{-3}
G	0.3127	0.3130	0.3217×10^{-3}	0.3127	0.3223×10^{-3}
H	0.5000	0.5000	0.1438×10^{-3}	0.4998	0.1479×10^{-3}

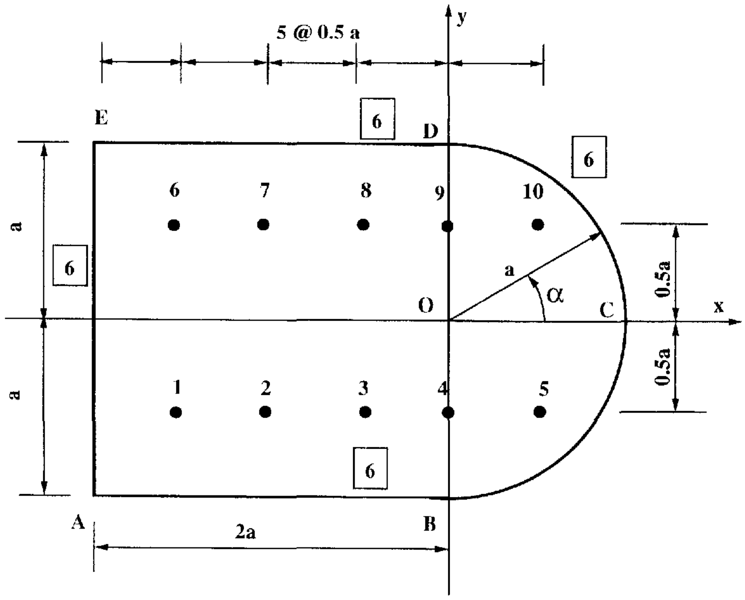

Fig. 4. Plate geometry used for the study of spatial variability of the material parameter.

where c_1 and c_2 are the correlation lengths, σ^2 is the variance of the random field, and **x** and **y** are two points in the interior of the domain with respective coordinates (x_1, x_2) and (y_1, y_2). The numerical values for the parameters that describe the random field are taken as $c_1 = 6.0$, $c_2 = 4.0$, $\bar{k} = 1.0$ and $\sigma = 0.14$. The discretization of the random field is performed using eqn (37), for which the interpolation functions are defined in eqns (30) and (38). The total number of random variables used to discretize the random field is $n = n_e + n_p$, where n_e is the number of boundary elements used to discretize the contour of the domain and n_p is the number of internal points for which the response statistics are required. The boundary element discretization, together with the locations of internal points, are shown in Fig. 4. For this problem, $n_e = 24$ and $n_p = 10$, resulting in a total of $n = 34$ random variables. It is seen from eqn (44) that the discretization of the random field results in a set of correlated random variables. This set is transformed into an uncorrelated set and the analysis is performed by retaining different numbers, n_u, of the uncorrelated variables. The response statistics of the internal potentials at the locations 1–10 are given in Table 8. The results of Monte Carlo simulations using 5000 samples are also given in Table 8. The random variables for the Monte Carlo simulations are generated in two steps as in Kaljević and Saigal (1993). It is seen from Table 8 that a good agreement of results is achieved

Table 8. Analysis of potential flow through a plate geometry with spatially random material parameter

		Mathematical expectations				Variances			
		This study				This study			
Loc.	Deterministic solution	5 RV	10 RV	20 RV	Monte Carlo simulation	5 RV	10 RV	20 RV	Monte Carlo simulation
1	1.9546	1.9800	1.9800	1.9813	1.9818	0.0287	0.0502	0.0570	0.0623
2	1.9128	1.9369	1.9380	1.9392	1.9412	0.0254	0.0321	0.0366	0.0375
3	1.8087	1.8305	1.8321	1.8335	1.8388	0.0182	0.0197	0.0200	0.0218
4	1.5541	1.5716	1.5732	1.5749	1.5801	0.0116	0.0121	0.0124	0.0142
5	0.9930	1.0019	1.0030	1.0048	1.0083	0.0035	0.0039	0.0047	0.0052
6	0.9554	0.9810	0.9810	0.9821	0.9865	0.0293	0.0503	0.0581	0.0620
7	0.9167	0.9407	0.9419	0.9431	0.9477	0.0262	0.0328	0.0346	0.0389
8	0.8276	0.8494	0.8511	0.8524	0.8585	0.0186	0.0202	0.0204	0.0244
9	0.6372	0.6548	0.6564	0.6579	0.6629	0.0120	0.0125	0.0128	0.0147
10	0.2185	0.2275	0.2285	0.2303	0.2333	0.0036	0.0041	0.0049	0.0054

with a relatively small number of random variables retained. A fast convergence of the computations with the number of random variables retained is also seen from Table 8.

7. CONCLUSIONS

The problems of stochastic potential flow through homogeneous domains that are characterized by a set of random parameters were analysed in this study. The probabilistic boundary element formulation, originally developed for the analysis of two-dimensional problems in elastostatics, was extended for the treatment of these problems. The random operator of the governing differential equation was described by a set of correlated random variables that may represent either a random geometric configuration or a random material parameter of the domain. The random geometric configuration was modeled by a finite set of correlated random variables while the random material parameter was modeled as a homogeneous random field with a constant mathematical expectation over the domain. The random field was first discretized into a finite set of correlated random variables and then the general procedure was applied. The transformation of the correlated random variables into an uncorrelated set was performed to reduce the number of operations. The response statistics obtained from the present formulations were compared with those obtained from Monte Carlo simulations using 5000 samples and a good agreement between results was observed.

The response statistics of internal potentials were also calculated for both cases of randomness. These calculations required the modeling of the interior of the domain being analysed. For domains with random geometric configuration, the positions of the internal points were expressed in terms of parameters that define the random contour. Several models of the random interior were analysed including: (a) deterministic interior, (b) partially random interior and (c) fully random interior. It was observed from the numerical results that the assumptions on the randomness of the interior may significantly influence the statistics of internal potentials. It is noted, however, that these assumptions do not affect the results for boundary variables. For the case of random material parameter, additional variables were introduced to represent the interior of the domain. The response statistics of the contour variables are not affected by these additional variables if the original (nontransformed) random variables are used in the calculations. Variances of transformed random variables, however, depend on the number of random variables defined in the interior. This causes differences in the calculations of values of response variables at the contour of the object, when only a part of the transformed set is retained in the analysis. This difference is pronounced only if a very small number of the transformed variables is retained, and becomes negligible for a larger number of variables retained.

An assessment of the influence of individual random variables on the response statistics was also done using numerical data. It was observed that for the circular contour, the radius has the most significant effect on the response, while the effect of the randomness of the coordinates of the center is negligible. The range of application of the present formulations was investigated, and it was seen from the example problems considered here that satisfactory results are obtained for a value of the coefficient of variation of the random parameter of up to 20%.

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APPENDIX

Detailed expressions for the derivatives of boundary element kernels and the particular integral expressions for distributed sources that are considered in this study are given below.

(a) Kernel G:

$$G(\mathbf{x},\xi)_{,l} = -\frac{1}{2\pi}\frac{1}{r}r_{,l} \tag{A1}$$

$$G(\mathbf{x},\xi)_{,lm} = -\frac{1}{2\pi}\left(-\frac{1}{r^2}r_{,l}r_{,m}+\frac{1}{r}r_{,lm}\right) \tag{A2}$$

(b) Kernel F:

$$F(\mathbf{x},\xi)_{,l} = -\frac{1}{2\pi}\left[-\frac{2}{r^3}r_{,l}y_sn_s+\frac{1}{r^2}(y_{s,l}n_s+y_sn_{s,l})\right] \tag{A3}$$

$$F(\mathbf{x},\xi)_{,lm} = -\frac{1}{2\pi}\left[\left(\frac{6}{r^4}r_{,l}r_{,m}-\frac{2}{r^3}r_{,lm}\right)y_sn_s-\frac{2}{r^3}r_{,l}(y_{s,m}n_s+y_sn_{s,m})-\frac{2}{r^3}r_{,m}(y_{s,l}n_s+y_sn_{s,l})\right.$$
$$\left.+\frac{1}{r^2}(y_{s,lm}n_s+y_{s,l}n_{s,m}+y_{s,m}n_{s,l}+y_sn_{s,lm})\right]. \tag{A4}$$

In eqns (A1)–(A4), the comma denotes differentiation with respect to random variables; l and m denote the random variables b_l and b_m, respectively; repeated indices denote summation; and the quantities r, n_s and y_s, $s = 1, 2$, were defined in the paragraph following eqn (3). The derivatives of the radius, r, and its projections on the coordinate axes, y_s, as well as the derivatives of the Jacobian, J, with respect to the random variables were given by Kaljević and Saigal (1993) and are not repeated here.

(c) Particular integrals:

(i) $\Psi(x_1, x_2) = \Psi_0$

$$\Phi^p = -\frac{\Psi_0}{4k}x_sx_s \quad \Phi^p_n = -\frac{\Psi_0}{2k}x_sn_s \tag{A5}$$

$$\Phi^p_{,l} = -\frac{\Psi_0}{2k} x_s x_{s,l} \tag{A6a}$$

$$\Phi^p_{n,l} = -\frac{\Psi_0}{2k} (x_{s,l} n_s + x_s n_{s,l}) \tag{A6b}$$

$$\Phi^p_{,lm} = -\frac{\Psi_0}{2k} (x_{s,l} x_{s,m} + x_s x_{s,lm}) \tag{A6c}$$

$$\Phi^p_{n,lm} = -\frac{\Psi_0}{2k} (x_{s,lm} n_s + x_{s,l} n_{s,m} + x_{s,m} n_{s,l} + x_s n_{s,lm}) \tag{A6d}$$

(ii) $\Psi(x_1, x_2) = \frac{\Psi_0}{a^2}(x_1^2 + x_2^2)$

$$\Phi^p = -\frac{\Psi_0}{12ka^2}(x_1^4 + x_2^4) \quad \Phi^p_n = -\frac{\Psi_0}{3ka^2}(x_1^3 n_1 + x_2^3 n_2) \tag{A7}$$

$$\Phi^p_{,l} = -\frac{\Psi_0}{3ka^2}(x_1^3 x_{1,l} + x_2^3 x_{2,l}) \tag{A8a}$$

$$\Phi^p_{n,l} = -\frac{\Psi_0}{3ka^2}(3x_1^2 n_1 x_{1,l} + 3x_2^2 n_2 x_{2,l}) \tag{A8b}$$

$$\Phi^p_{,lm} = -\frac{\Psi_0}{3ka^2}(3x_1^2 x_{1,i} x_{1,j} + x^3 x_{1,ij} + 3x_2^2 x_{2,i} x_{2,j} + x^3 x_{2,ij}) \tag{A8c}$$

$$\Phi^p_{n,lm} = -\frac{\Psi_0}{3ka^2}[6x_1 n_1 x_{1,i} x_{1,j} + 3x_1^2(n_{1,j} x_{1,i} + n_{1,i} x_{1,j} + n_1 x_{1,ij}) + x_1^3 n_{1,ij}$$

$$+ 6x_2 n_2 x_{2,i} x_{2,j} + 3x_2^2(n_{2,j} x_{2,i} + n_{2,i} x_{2,j} + n_2 x_{2,ij}) + x_2^3 n_{2,ij}] \tag{A8d}$$

(iii) $\Psi(x_1, x_2) = \frac{\Psi_0}{ab} x_1 x_2$

$$\Phi^p = -\frac{\Psi_0}{12k}\frac{1}{ab} x_1 x_2 (x_1^2 + x_2^2) \quad \Phi^p_n = -\frac{\Psi_0}{12k}\frac{1}{ab}[x_2(3x_1^2 + x_2^2) n_1 + x_1(3x_2^2 + x_1^2) n_2] \tag{A9}$$

$$\Phi^p_{,l} = -\frac{\Psi_0}{12k}\frac{1}{ab}[x_2(3x_1^2 + x_2^2) x_{1,l} + x_1(3x_2^2 + x_1^2) x_{2,l}] \tag{A10a}$$

$$\Phi^p_{n,l} = -\frac{\Psi_0}{12k}\frac{1}{ab}\{[6x_1 x_2 x_{1,l} + 3(x_1^2 + x_2^2) x_{2,l}] n_1 + x_2(3x_1^2 + x_2^2) n_{1,l}$$

$$+ [3(x_1^2 + x_2^2) x_{1,l} + 6x_1 x_2 x_{2,l}] n_2 + x_1(x_1^2 + 3x_2^2) n_{2,l}\} \tag{A10b}$$

$$\Phi^p_{,lm} = -\frac{\Psi_0}{12k}\frac{1}{ab}[(3x_1^2 + x_2^2)(x_{1,l} x_{2,m} + x_2 x_{1,lm}) + 2x_2(3x_1 x_{1,m} + x_2 x_{2,m}) x_{1,l}$$

$$+ (x_1^2 + 3x_2^2)(x_{1,m} x_{2,l} + x_1 x_{2,lm}) + 2x_1(x_1 x_{1,m} + 3x_2 x_{2,m}) x_1 x_{2,l}] \tag{A10c}$$

$$\Phi^p_{n,lm} = -\frac{\Psi_0}{12k}\frac{1}{ab}\{[6(x_2 x_{1,l} x_{2,m} + x_1 x_{1,l} x_{2,m} + x_1 x_2 x_{1,lm})$$

$$+ 6(x_1 x_{1,m} + x_2 x_{2,m}) x_{2,l} + 3(x_1^2 + x_2^2) x_{2,lm}] n_1 + [6x_1 x_2 x_{1,l} + 3(x_1^2 + x_2^2) x_{2,l}] n_{1,m}$$

$$+ [(3x_1^2 + x_2^2) x_{2,m} + 2(3x_1 x_{1,m} + x_2 x_{2,m})] n_{1,l} + x_2(3x_1^2 + x_2^2) n_{1,lm}$$

$$+ [6(x_2 x_{1,l} x_{2,m} + x_1 x_{1,l} x_{2,m} + x_1 x_2 x_{1,lm}) + 6(x_1 x_{1,m} + x_2 x_{2,m}) x_{2,l} + 3(x_1^2 + x_2^2) x_{2,lm}] n_1$$

$$+ [6x_1 x_2 x_{1,l} + 3(x_1^2 + x_2^2) x_{2,l}] n_{1,m} + [(3x_1^2 + x_2^2) x_{2,m} + 2(3x_1 x_{1,m} + x_2 x_{2,m})] n_{1,l} + x_2(3x_1^2 + x_2^2) n_{1,lm}\}. \tag{A10d}$$

In eqns (A5)–(A10), Φ^p denotes the particular integral for the potential, Φ^p_n is the corresponding normal derivative of the particular integral, Ψ_0 represents the reference intensity of a distributed source, a and b are the normalization lengths, and x_1 and x_2 represent the coordinates of the point $\mathbf{x}$ in the domain Ω.